AN ADDEDUM TO THE PAPER "SOME ELEMENTARY ESTIMATES FOR THE NAVIER-STOKES SYSTEM"

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Abstract. In this paper we give a proof of the existence of global regular solutions to the Fourier transformed Navier-Stokes system with small initial data in $\Phi(2)$ via an iteration argument. The proof of the regularity theorem is a minor modification of the proof given in the paper "Some elementary estimates for the Navier-Stokes system", so this paper is intended to be just a complement to the afore mentioned paper.

1. Introduction

A Generalized Navier-Stokes system (with periodic boundary conditions on $[0,1]^3$) is a system of the form

$$(1)\quad \begin{aligned} v^k(\xi,t) &= \psi^k(\xi)\exp\left(-|\xi|^2 t\right) \\ &\quad + \int_0^t \exp\left(-|\xi|^2(t-s)\right)\sum_{\mathbf{q}\in\mathbb{Z}^3} M_{ijk}(\xi)\, v^i(q,s)\, v^j(\xi-q,s)\, ds, \end{aligned}$$

for $\xi\in\mathbb{Z}^3$, and where $M_{ijk}(\xi)$ satisfies the bound

$$|M_{ijk}(\xi)|\leq|\xi|.$$

To solve this problem it is usual to consider the following iteration scheme

$$\begin{aligned} v^k_{n+1}(\xi,t) &= \psi^k(\xi)\exp\left(-|\xi|^2 t\right) \\ &\quad + \int_0^t \exp\left(-|\xi|^2(t-s)\right)\sum_{\mathbf{q}\in\mathbb{Z}^3} M_{ijk}(\xi)\, v^i_n(q,s)\, v^j_n(\xi-q,s)\, ds. \end{aligned}$$

In what follows we will show the convergence of this method for small initial conditions on $\Phi(2)$ (for the definition of the space $\Phi(2)$ see [4]). More exactly we will show that

Theorem 1. *There exists an $\epsilon>0$ such that if $\|\psi\|_2<\epsilon$, then (1) has a global regular solution with initial condition ψ.*

The main purpose on writing this note is for it to serve as a complement to our paper [4], and to show that the free divergence condition, neither the fact of considering Leray-Hopf weak solutions is an issue for the proofs presented in that paper.

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2. Existence

We start with two auxiliary results,

Lemma 1. *There exists an $\epsilon > 0$ such that if $\|\psi\| < \epsilon$ then the sequence $v_n^k(\xi,t)$ is uniformly bounded on $[0,T]$ for ξ fixed.*

Proof. To proof this fact it is enough to show that if

$$\left|v_n^k(\xi,t)\right| \leq \frac{\epsilon}{|\xi|^2}$$

then

$$\left|\sum_{\mathbf{q}\in\mathbb{Z}^3} M_{ijk}(\xi)\, v_n^i(q,s)\, v_n^j(\xi-q,s)\right| \leq c\epsilon^2, \tag{2}$$

where c is a universal constant, because then we would have, for any $t \geq 0$ and $\epsilon > 0$ small enough,

$$\begin{aligned}
\left|v_{n+1}^k(\xi,t)\right| &\leq \left|\psi^k(\xi)\right| \exp\left(-|\xi|^2 t\right) + c\int_0^t \exp(t-s)\,\epsilon^2\, ds \\
&\leq \frac{\epsilon}{|\xi|^2}\exp(-|\xi|^2 t) + \frac{c\epsilon^2}{|\xi|^2}\left(1-\exp\left(-|\xi|^2 t\right)\right) \\
&\leq \frac{\epsilon}{|\xi|^2}\exp(-|\xi|^2 t) + \frac{\epsilon}{|\xi|^2}\left(1-\exp\left(-|\xi|^2 t\right)\right) = \frac{\epsilon}{|\xi|^2}.
\end{aligned}$$

We proceed to show the validity of (2). Write

$$\sum_{\mathbf{q}\in\mathbb{Z}^3} M_{ijk}(\xi)\, v_n^i(q,s)\, v_n^j(\xi-q,s) = I + II + III,$$

where

$$\begin{aligned}
I &= \sum_{1\leq|q|\leq 2|\xi|,\,1\leq|\xi-q|\leq\frac{|\xi|}{2}} M_{ijk}(\xi)\, v_n^i(\xi,t)\, v_n^j(\xi-q,t), \\
II &= \sum_{1\leq|q|\leq 2|\xi|,\,|\xi-q|>\frac{|\xi|}{2}} M_{ijk}(\xi)\, v_n^i(\xi,t)\, v_n^j(\xi-q,t),
\end{aligned}$$

and

$$III = \sum_{|q|>2|\xi|} M_{ijk}(\xi)\, v_n^i(\xi,t)\, v_n^j(\xi-q,t).$$

To estimate I observe that if $|\xi-q| \leq \frac{|\xi|}{2}$, then $|q| \geq \frac{|\xi|}{2}$. Therefore, using that

$$|M_{ijk}(\xi)| \leq c\,|\xi|$$

and the elementary inequality

$$\sum_{1\leq|q|<r} \frac{1}{|q|^2} \leq cr \tag{3}$$

(where c is a universal constant) we can bound as follows,

$$\begin{aligned}
|I| &\leq c\,|\xi|\,\frac{\epsilon^2}{|\xi|^2} \sum_{1\leq|\xi-q|\leq\frac{|\xi|}{2}} \frac{1}{|\xi-q|^2} \\
&\leq \frac{c\epsilon^2}{|\xi|}\frac{|\xi|}{2} = c\epsilon^2.
\end{aligned}$$

II can be estimated in the same way, so we also obtain

$$|II| \leq c\epsilon^2.$$

To estimate III, first notice that $|q| > 2|\xi|$ implies that $|\xi - q| \geq \frac{1}{2}|q|$. Hence, using the inequality

$$\sum_{|q|\geq r} \frac{1}{|q|^4} \leq \frac{c}{r}, \tag{4}$$

we can bound as follows,

$$\begin{aligned} |III| &\leq c|\xi|\epsilon^2 \sum_{|q|>2|\xi|} \frac{1}{|q|^2}\frac{1}{|\xi-q|^2} \\ &\leq c|\xi|\epsilon^2 \sum_{|q|>2|\xi|} \frac{1}{|q|^4} \\ &\leq c|\xi|\epsilon^2 \frac{1}{|\xi|} = c\epsilon^2. \end{aligned}$$

This shows the lemma. □

Lemma 2. *If there is an* $\epsilon > 0$ *such that the sequence* $v_n^k(\xi, t)$ *satisfies*

$$\left\|v_n^k(t)\right\|_2 < \epsilon \quad \textit{for all} \quad t \in [0, T]$$

The sequence $v_n^k(\xi, t)$ *is equicontinuous on* $[0, T]$ *for* ξ *fixed.*

Proof. Let $t_1, t_2 \in (\rho, T)$, $t_2 > t_1$.Then we estimate for ξ fixed

$$\left|v_{n+1}^k(\xi, t_2) - v_{n+1}^k(\xi, t_1)\right| \leq I + II + III$$

where

$$I = \left|\psi^k(\xi)\right| \left|\exp\left(-|\xi|^2 t_2\right) - \exp\left(-|\xi|^2 t_1\right)\right|,$$

$$\begin{aligned} II &= \int_0^{t_1} \left|\exp\left(-|\xi|^2(t_2 - s)\right) - \exp\left(-|\xi|^2(t_1 - s)\right)\right| \\ &\quad \sum_{\mathbf{q}\in\mathbb{Z}^3} \left|M_{ijk}(\xi)\, v_n^i(q, s)\, v_n^j(\xi - q, s)\right| ds \end{aligned}$$

and

$$III = \int_{t_1}^{t_2} \exp\left(-|\xi|^2(t - s)\right) \sum_{\mathbf{q}\in\mathbb{Z}^3} \left|M_{ijk}(\xi)\, v_n^i(q, s)\, v_n^j(\xi - q, s)\right| ds.$$

Let us bound each of the previous expressions,

$$\begin{aligned} I &= \left|\exp\left(-|\xi|^2 t_1\right)\right| \left|1 - \exp\left(-|\xi|^2(t_2 - t_1)\right)\right| \\ &\leq \frac{\epsilon}{|\xi|^2}|\xi|^2|t_2 - t_1| = \epsilon|t_2 - t_1|, \end{aligned}$$

$$\begin{aligned}
II &\leq \int_0^{t_1} \left|\exp\left(-|\xi|^2(t_2-s)\right)-\exp\left(-|\xi|^2(t_1-s)\right)\right|\epsilon^2\,ds \\
&= \int_{\tau_n}^{t_1} \exp\left(-|\xi|^2(t_1-s)\right)\left|1-\exp\left(-|\xi|^2(t_2-t_1)\right)\right|\epsilon^2\,ds \\
&\leq |\xi|^2(t_2-t_1)\frac{1}{|\xi|^2}\left|1-\exp\left(-|\xi|^2 t_1\right)\right|,
\end{aligned}$$

$$\begin{aligned}
III &\leq \int_{t_1}^{t_2} \epsilon^2\exp\left(-|\xi|^2(t_2-s)\right)\,ds \\
&\leq \frac{\epsilon^2}{|\xi|^2}\left|1-\exp\left(-|\xi|^2(t_2-t_1)\right)\right| \leq \frac{1}{|\xi|^2}\epsilon^2|\xi|^2|t_2-t_1|,
\end{aligned}$$

and hence

$$\left|v_{n+1}^k(\xi,t_2)-v_{n+1}^k(\xi,t_1)\right| < C(\epsilon)(t_2-t_1)$$

for $n \geq 0$, and the lemma is proved. □

The previous Lemmas via the theorem of Arzela-Ascoli, using Cantor's diagonal procedure, show that there is a well defined $v \in \Phi(2)$ defined on $[0,T]$ such that,

$$\begin{aligned}
(5)\quad v^k(\xi,t) &= v^k(\xi,t)\exp\left(-|\xi|^2 t\right) \\
&+ \int_0^t \exp\left(-|\xi|^2(t-s)\right)\sum_{\mathbf{q}\in\mathbb{Z}^3} M_{ijk}(\xi)\,v^i(q,s)\,v^j(\xi-q,s)\,ds
\end{aligned}$$

Let us give a proof of this. To simplify notation, let us assume that the sequence converging uniformly on $[0,T]$ for each ξ is the sequence $v_n(\xi,t)$. By what we have shown, there exists a D not depending on t, ξ or n such that

$$\left|v_n^j(\xi,t)\right| \leq \frac{D}{|\xi|^2}.$$

Let ξ be fixed, and let $\eta > 0$ arbitrary. the previous estimate allows us to choose a Q such that

$$\left|\sum_{|q|\geq Q} M_{ijk}(\xi)\,v_n^i(\xi,t)\,v_n^j(\xi,t)\right| \leq \eta.$$

and also that the same inequality is valid with v_n replaced by v (this can be done since the choice of Q only depends on D). Hence we have

$$\begin{aligned}
\Big|v_{n+1}^k(\xi,t) &- \psi^k(\xi)\exp\left(-|\xi|^2 t\right) \\
&- \int_0^t \exp\left(-|\xi|^2(t-s)\right)\sum_{1\leq|q|<Q} M_{ijk}(\xi)\,v_n^i(q,s)\,v_n^j(\xi-q,s)\,ds\Big| \\
&\leq \eta
\end{aligned}$$

Taking $n \to \infty$, we obtain

$$\begin{aligned} \left| v^k(\xi,t) \right. & - \psi^k(\xi)\exp\left(-|\xi|^2 t\right) \\ & - \left. \int_0^t \exp\left(-|\xi|^2(t-s)\right) \sum_{1\le|q|<Q} M_{ijk}(\xi)\, v^i(q,s)\, v^j(\xi-q,s)\, ds \right| \\ & \le \eta \end{aligned}$$

and from this follows that

$$\begin{aligned} \left| v^k(\xi,t) \right. & - \psi^k(\xi)\exp\left(-|\xi|^2 t\right) \\ & - \left. \int_0^t \exp\left(-|\xi|^2(t-s)\right) \sum_{q\in\mathbb{Z}^3} M_{ijk}(\xi)\, v^i(q,s)\, v^j(\xi-q,s)\, ds \right| \\ & \le 2\eta. \end{aligned}$$

Since $\eta > 0$ is arbitrary, our claim is proved.

3. Regularity

We shall show now that the solutions produced by the iteration scheme are regular under certain smallness condition. Indeed, we have

Theorem 2. *Let $v \in L^\infty(0,T;\Phi(2))$ be a solution to (1). There exists an $\epsilon > 0$ such that if there is a k_{-1} for which v satisfies*

$$\sup_{|\xi|\ge k_{-1}} |\xi|^2 \left| v^k(\xi,t) \right| < \epsilon \quad \textit{for all} \quad t \in (0,T) \tag{6}$$

then v is smooth.

To prove Theorem 2 we will need to estimate term

$$\sum M_{ijk}(\xi)\, u^i(q)\, u^j(\xi-q)\,.$$

This is the content of Lemma 3. But before we state and prove Lemma 3 and in order to express our estimates in a convenient way we will define to sequences of numbers. Namely

$$\begin{cases} \mu_0 = 1 \quad \mu_1 = 1 \\ \mu_{n+1} = 2\mu_n - 1, \quad n \ge 2 \end{cases}$$

and

$$k_n = \frac{1}{\epsilon^{2^n}} k_0$$

where k_0 is such that

$$\frac{k_{-1}}{k_0} \cdot D < \min\left\{\epsilon, \frac{1}{2}\right\}$$

and $D = \sup_{(0,T)} \|u(t)\|$.

We are now ready to estate and prove,

Lemma 3. *Assume that for all ξ such that $|\xi| \ge k_{-1}$*

$$\left| v^k(\xi,s) \right| \le \frac{\epsilon}{|\xi|^2}$$

and if $|\xi| \geq k_m$

$$\left|v^k(\xi,s)\right| \leq \frac{\epsilon^{\mu_m}}{|\xi|^2}$$

Then for $|\xi| \geq k_{m+1}$ *it holds that,*

$$\left|\sum_{q\in\mathbb{Z}^3} M_{ijk}(\xi)\, v^i(q,s)\, v^j(\xi-q,s)\right| \leq \epsilon^{\mu_{m+1}}.$$

Proof. First recall that $|M_{ijk}(\xi)| \leq c\,|\xi|$.

$$\begin{aligned}(7)\; I \quad &\leq \quad |\xi| \sum_{1\leq|q|<k_{-1}} \left|v^i(q,s)\, v^j(\xi-q,s)\right| + |\xi| \sum_{k_{-1}\leq|q|<k_m} \left|v^i(q,s)\, v^j(\xi-q,s)\right| \\ &\quad + |\xi| \sum_{|q|\geq k_m} \left|v^i(q,s)\, v^j(\xi-q,s)\right|\end{aligned}$$

We estimate the first sum. Observe that $k_{-1} \leq \frac{|\xi|}{2}$, so if $|q| < k_{-1}$, we must have $|\xi - q| \geq \frac{|\xi|}{2}$. Hence, using the elementary inequality (3), we can bound

$$\begin{aligned}\sum_{1\leq|q|<k_{-1}} \left|v^i(q,s)\, v^j(\xi-q,s)\right| \quad &\leq \quad \frac{4\epsilon^{\mu_m}}{|\xi|^2} \sum_{1\leq|q|<k_{-1}} \frac{D}{|q|^2} \\ &\leq \quad 4c\epsilon^{\mu_m} \frac{k_{-1}}{k_m} \leq 4c\epsilon^{2\mu_m}\end{aligned}$$

To estimate the second sum, notice that if $|\xi| \geq k_{m+1}$ and $|q| \leq k_m$, then $|\xi - q| \geq \frac{|\xi|}{2} \geq k_m$. All this said, using inequality (3) again we obtain,

$$\begin{aligned}\sum_{1\leq|q|<k_m} \left|v^i(q,s)\, v^j(\xi-q,s)\right| \quad &\leq \quad \frac{4\epsilon^{\mu_m}}{|\xi|^2} \sum_{1\leq|q|<k_m} \frac{\epsilon}{|q|^2} \\ &\leq \quad \frac{4\epsilon^{\mu_m}}{|\xi|^2}\epsilon k_m\end{aligned}$$

Observe now that $\frac{k_m}{k_{m+1}} \leq \epsilon^{2^m} \leq \epsilon^{\mu_m}$. This yields the bound,

$$\sum_{1\leq|q|<k_m} \left|v^i(q,s)\, v^j(\xi-q,s)\right| \leq \frac{4\epsilon^{\mu_m}}{|\xi|} \frac{k_m}{k_{m+1}} \leq \frac{4\epsilon^{2\mu_m}}{|\xi|}$$

To estimate the second sum on the righthanside of (7) we split it into three sums, namely

$$\begin{aligned}(8) \qquad \sum_{|q|\geq k_m} \left|v^i(q,s)\, v^j(\xi-q,s)\right| \quad &= \quad \sum_{k_m\leq|q|<\frac{|\xi|}{2}} \left|v^i(q,s)\, v^j(\xi-q,s)\right| \\ &\quad + \sum_{\frac{|\xi|}{2}\leq|q|<2|\xi|} \left|v^i(q,s)\, v^j(\xi-q,s)\right| \\ &\quad + \sum_{|q|\geq 2|\xi|} \left|v^i(q,s)\, v^j(\xi-q,s)\right|\end{aligned}$$

Estimating the three sums on the right hand side separately. Observe that if $|q| \leq \frac{|\xi|}{2}$ then we must have $|\xi - q| \geq \frac{|\xi|}{2} > k_m$. Therefore, using inequality (3), we get

$$\sum_{k_m \le |q| < \frac{|\xi|}{2}} \left| v^i(q,s)\, v^j(\xi - q, s) \right| \le \frac{4\epsilon^{2\mu_m}}{|\xi|^2} \sum_{1 \le |q| < \frac{|\xi|}{2}} \frac{1}{|q|^2} \le \frac{4\epsilon^{2\mu_m}}{|\xi|}$$

To estimate the second sum we split it into two sums,

$$\sum_{\frac{|\xi|}{2} \le |q| < 2|\xi|} \left| v^i(q,s)\, v^j(\xi - q, s) \right| = \sum_{\frac{|\xi|}{2} \le |q| < 2|\xi|, k_m \le |\xi - q|} \left| v^i(q,s)\, v^j(\xi - q, s) \right| + \sum_{\frac{|\xi|}{2} \le |q| < 2|\xi|, |\xi - q| < k_m} \left| v^i(q,s)\, v^j(\xi - q, s) \right|$$

Estimating the first sum on the righthandside of the previous equality,

$$\sum_{\frac{|\xi|}{2} \le |q| < 2|\xi|, |\xi - q| \ge k_m} \left| v^i(q,s)\, v^j(\xi - q, s) \right| \le \frac{4\epsilon^{2\mu_m}}{|\xi|^2} \sum_{1 \le |\xi - q| < 3|\xi|} \frac{1}{|\xi - q|^2} \le \frac{12\epsilon^{2\mu_m}}{|\xi|}$$

The estimation of the second sum proceeds in exactly the same way as the estimation of the first sum on the right hand side of (7), and hence we obtain

$$\sum_{\frac{|\xi|}{2} \le |q| < 2|\xi|, |\xi - q| < k_m} \left| v^i(q,s)\, v^j(\xi - q, s) \right| \le \frac{4\epsilon^{2\mu_m}}{|\xi|}.$$

Now we estimate the third sum in the righthandside of (8). Using that $|q| \ge 2|\xi|$ implies that $|\xi - q| \ge \frac{1}{2}|q|$, and inequality (4) we can bound,

$$\sum_{|q| \ge 2|\xi|} \left| v^i(q,s)\, v^j(\xi - q, s) \right| \le 4\epsilon^{2\mu_m} \sum_{|q| \ge 2|\xi|} \frac{1}{|q|^4} \le \frac{4\epsilon^{2\mu_m}}{|\xi|}$$

Putting all the previous estimations together, we arrive at

$$\left| \sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi)\, v^i(q,s)\, v^j(\xi - q, s) \right| \le |\xi| \left(\frac{28\epsilon^{\mu_m}}{|\xi|} \right),$$

and if we assume $0 < \epsilon < \frac{1}{28}$, the previous inequality reads as

$$\left| \sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi)\, v^i(q,s)\, v^j(\xi - q, s) \right| \le \epsilon^{2\mu_m - 1} \le \epsilon^{\mu_{m+1}},$$

and the Lemma is proved. □

3.1. **Proof of Theorem 2.** Given $0 < \rho < T$, we will first show that for a constant $K(\rho)$, there exists a constant D such that if $|\xi| \ge K(\rho)$ then

$$\left| v^k(\xi, t) \right| \le \frac{D}{|\xi|^{2+\frac{1}{4}}} \quad \text{if} \quad t > \rho.$$

Define

$$\tau_m = \rho - \frac{\rho}{2^m}.$$

We will show by induction that

$$v^k(\xi,t) \le \frac{\epsilon^{\mu_n}}{|\xi|^2} \quad \text{if} \quad t > \tau_n \quad \text{and} \quad |\xi| \ge k_n. \tag{P}$$

For $n = 0$, our choice of k_0 guarantees that (P) holds. Assume that (P) holds for $n = m$. First observe that v satisfies

$$\begin{aligned} v^k(\xi,t) &= v^k(\xi,\tau_n)\exp\left(-|\xi|^2(t-\tau_n)\right) \\ &\quad + \int_{\tau_n}^t \exp\left(-|\xi|^2(t-s)\right)\sum_{\mathbf{q}\in\mathbb{Z}^3} M_{ijk}(\xi)\, v^i(q,s)\, v^j(\xi-q,s)\, ds. \end{aligned}$$

Using this identity, we bound as follows,

$$\begin{aligned} v^k(\xi,t) &\le v^k(\xi,\tau_m)\exp\left(-|\xi|^2(t-\tau_m)\right) + \int_{\tau_m}^t \exp\left(-|\xi|^2(t-s)\right)\epsilon^{2\mu_m}\, ds \\ &\le \frac{\epsilon^{\mu_m}}{|\xi|^2}\exp\left(-k_{m+1}(\tau_{m+1}-\tau_m)\right) \\ &\quad + \frac{\epsilon^{2\mu_m}}{|\xi|^2}\left(\exp\left(-|\xi|^2\tau_m\right) - \exp\left(-|\xi|^2 t\right)\right) \\ &\le \frac{\epsilon^{\mu_m}}{|\xi|^2} + \frac{\epsilon^{2\mu_m}}{|\xi|^2}. \end{aligned}$$

From this last bound it follows that if $t \ge \rho > \rho - \frac{\rho}{2^m}$, then if $k_m \le |\xi| < k_{m+1}$ it holds that

$$\left|v^k(\xi,t)\right| \le \frac{\epsilon^{\mu_m}}{|\xi|^2}.$$

Since $\mu_m \ge 2^{n-1}$ and $k_m = \frac{k_0}{\epsilon^{2^m}}$, it is easy to check that $\epsilon^{\mu_m} \le \frac{k_0^{\frac{1}{4}}}{|\xi|^{\frac{1}{4}}}$. Hence for all $t \ge \rho$ the following estimate holds,

$$\left|v^k(\xi,t)\right| \le \frac{D}{|\xi|^{2+\frac{1}{4}}}.$$

The following Lemma will then finish the proof of Theorem 2.

Lemma 4. *Let v be a solution to (FNS) such that for all $t \in (0,T)$ satisfies*

$$\left|v^k(\xi,t)\right| \le \frac{D}{|\xi|^{2+\eta}}$$

with D and $\eta > 0$ independent of t. Then v is smooth.

Proof. Let $\rho > 0$. Under the hypothesis of the Lemma, we will show that there exists a constant $K := K(\rho)$ such that if $t > T$ and $|\xi| > K$, then for a constant E independent of time,

$$\left|v^k(\xi,t)\right| \le \frac{E}{|\xi|^{2+\min\left(\frac{1}{2},\frac{3}{2}\eta\right)}}.$$

Since $\rho > 0$ is arbitrary, a finite number of applications of the previous claim shows that for any $\rho > 0$, the Fourier transform of v decays faster than any polynomial, and this shows the lemma.

First, we will estimate the term

$$S = \sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi)\, v^i(\xi, s)\, v^j(\xi, s)$$

under the hypotesis of the lemma. In order to do this we write,

$$S = I_a + I_b + II_a + II_b + III_a + III_b + IV_a + IV_b$$

where

$$I_a = \sum_{1 \le |q| \le \sqrt{|\xi|}} M_{ijk}(\xi)\, v^i(\xi, s)\, v^j(\xi, s),$$

$$II_a = \sum_{\sqrt{|\xi|} < q \le \frac{|\xi|}{2}} M_{ijk}(\xi)\, v^i(\xi, s)\, v^j(\xi, s),$$

$$III_a = \sum_{|q| \ge \frac{|\xi|}{2}, 1 \le |\xi - q| < 2|\xi|} M_{ijk}(\xi)\, v^i(\xi, s)\, v^j(\xi, s),$$

and

$$IV_a = \sum_{|q| \ge \frac{|\xi|}{2}, |\xi - q| \ge 2|\xi|} M_{ijk}(\xi)\, v^i(\xi, s)\, v^j(\xi, s)$$

The corresponding I_b, II_b, III_b and IV_b are the same as their a counterparts, except that the role of q and $\xi - q$ is interchanged. Noticed that by the triangular inequality not both q and $\xi - q$ can be less than $\frac{|\xi|}{2}$, and hence all possible cases are covered.

Since $|q| < \sqrt{|\xi|} < \frac{|\xi|}{2}$, and hence $|\xi - q| \ge \frac{|\xi|}{2}$. Hence we have,

$$\begin{aligned}
|I_a| &\le |\xi| \sum_{1 \le |q| \le \sqrt{|\xi|}} \frac{D}{|q|^{2+\eta}} \frac{D}{|\xi - q|^{2+\eta}} \\
&\le |\xi| \frac{2^{2+\eta} D^2}{|\xi|^{2+\eta}} \sum_{1 \le |q| \le \sqrt{|\xi|}} \frac{D}{|q|^2}
\end{aligned}$$

and by inequality (3)

$$\le |\xi| \frac{2^{2+\eta} D^2}{|\xi|^{2+\eta}} \sqrt{|\xi|} = \frac{2^{2+\eta} D^2}{|\xi|^{\frac{1}{2}+\eta}}.$$

Estimating II_a and III_a is pretty straightforward, via the inequality

$$\sum_{1 \le |q| < r} 1 \le c r^3.$$

Indeed,

$$\begin{aligned}
|II_a| &\le |\xi| \frac{2^{2+\eta} D}{|\xi|^{2+\eta}} \cdot \frac{D}{\left(\sqrt{|\xi|}\right)^{2+\eta}} \left(\sum_{|q| \le \frac{|\xi|}{2}} 1 \right) \\
&\le \frac{2^{2+\eta} D}{|\xi|^{1+\eta}} \cdot \frac{D}{|\xi|^{1+\frac{\eta}{2}}} |\xi|^3 = \frac{2^{2+\eta} D^2}{|\xi|^{\frac{3}{2}\eta}}.
\end{aligned}$$

$$\begin{aligned}
|III_a| &\leq |\xi| \sum_{|q|\geq\frac{|\xi|}{2},\frac{|\xi|}{2}\leq|\xi-q|<2|\xi|} \frac{D}{|q|^{2+\eta}}\frac{D}{|\xi-q|^{2+\eta}} \\
&\leq |\xi|\cdot\frac{2^{2+\eta}D^2}{|\xi|^{4+2\eta}}\left(\sum_{1\leq|\xi-q|<2|\xi|}1\right) \\
&\leq \frac{2^{2+\eta}}{|\xi|^{2\eta}}
\end{aligned}$$

Finally, using that $|\xi-q| \geq 2|\xi|$ and $|q| \geq \frac{|\xi|}{2}$ imply that $|q| \geq \frac{2}{3}|\xi-q|$ and inequality (4) we can bound IV_a as follows,

$$\begin{aligned}
|IV_a| &\leq |\xi| \sum_{|q|\geq\frac{|\xi|}{2},|\xi-q|\geq2|\xi|} \frac{D}{|q|^{2+\eta}}\frac{D}{|\xi-q|^{2+\eta}} \\
&\leq \frac{2^{2\eta}}{|\xi|^{2\eta}}\left(\frac{3}{2}\right)^{2+\eta}\sum_{|q|\geq\frac{|\xi|}{2}}\frac{D^2}{|q|^4} \\
&\leq |\xi|\frac{1}{|\xi|^{2\eta}}\frac{D^2}{|\xi|} = \frac{D^2}{|\xi|^{2\eta}}.
\end{aligned}$$

The proof is now complete. □

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